Introduction to quiver varieties

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The 49th Symposium on Ring Theory and Representation Theory

2016. 8. 31 ~ 9. 3 Osaka Prefecture University

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⟶ click Aug 31 – Sep. 3

Connecting representation theories of quivers and Lie algebras

Representations of quivers (preprojective algebras) $\xrightarrow{\text{moduli space}}$ quiver varieties $\xrightarrow{\text{homology groups + convolution product}}$ Representation of Lie algebras (Kac-Moody)

moduli space ≑ set of isomorphism classes

fields: noncommutative algebras; commutative algebraic geometry; Lie theory

non-noncommutative

tools: geometric invariant theory; topology of algebraic varieties

There are several other links.

- Ringel-Hall algebra $\longrightarrow$ realization of $U_q(n^-)$ quantum enveloping algebra; Lusztig's canonical base
- constructible functions on $\Lambda(V)$ (Lusztig)

motivating the above link

more recent development (motivated by earlier works of Lascoux-Leclerc-Thibon
Ariki)

- quiver Hecke algebra (Khovanov-Lauda, Rouquier)

quiver "$\longrightarrow$" a family of algebras A_V $\xrightarrow[\text{mod } A_V + (A_V, A_{V'})\text{-bimodule}]{}$ categorification of representation of Lie algebras

similar construction

$D^b\,\mathrm{Coh}(\text{quiver varieties}) \longrightarrow$ larger categorification ?

Notations

quiver $Q = (Q_0, Q_1)$ $\qquad o(h) \xrightarrow{h} i(h)$

$\overline{Q_1}$ = opposite arrows $\qquad \underset{\| \atop i(\bar h)}{o(h)} \xleftarrow{\bar h} \underset{\| \atop o(\bar h)}{i(h)}$

extend $\overline{}$ to $Q_1 \sqcup \overline{Q_1}$ by $\overline{\overline{h}} = h$

space of representations $V = \bigoplus_{i \in Q_0} V_i$: Q_0-graded vector space $/\mathbb{C}$

$$\mathbb{N}(V) := \bigoplus_{h \in Q_1} \mathrm{Hom}(V_{o(h)}, V_{i(h)}) \curvearrowleft G_V = \prod GL(V_i)$$

$\mathbb{N}(V)/G_V \xleftrightarrow{\text{bijective}}$ isomorphism classes of representations of Q with $\overrightarrow{\dim} = (\dim V_i)_{i \in Q_0}$

cotangent space

$$\mathbb{M}(V) = \mathbb{N}(V) \oplus \mathbb{N}(V)^* = \bigoplus_{h \in Q_1 \sqcup \overline{Q_1}} \mathrm{Hom}(V_{o(h)}, V_{i(h)})$$

moment map (preprojective alg.)

$$\mu : \mathbb{M}(V) \longrightarrow \bigoplus_i \mathrm{End}\, V_i \ ; \ (B_h)_{h \in Q_1 \sqcup \overline{Q_1}} \longmapsto \Big(\sum_{i(h)=i} \underset{\pm 1}{\underbrace{\varepsilon(h)}} B_h B_{\bar h} \Big)_i$$

Remark This has an origin in symplectic geometry

Lusztig's lagrangian $\Lambda(V) = \{ (B_h) \mid \mu = 0, \text{ nilpotent} \}$

<u>Fact</u> (Lusztig, Kashiwara-Saito) If $\cancel{\circlearrowleft}$, $\Lambda(V) \subset M(V)$ is lagrangian, and $\# \mathrm{Irr}\,\Lambda(V) = \dim U(n^-)_{wt = -\dim V}$

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framed representation of double quiver

V, W : Q_0-graded vector space $/\mathbb{C}$

new part

$$\mathbb{M}(V,W) = \bigoplus_{h \in Q_1 \sqcup \overline{Q_1}} \underbrace{\mathrm{Hom}(V_{o(h)}, V_{i(h)})}_{B_h} \oplus \bigoplus_{i \in Q_0} \underbrace{\mathrm{Hom}(W_i, V_i)}_{I_i} \oplus \underbrace{\mathrm{Hom}(V_i, W_i)}_{J_i}$$

Denote components by B_h, I_i, J_i

moment map

$$\mu : \mathbb{M}(V,W) \longrightarrow \bigoplus \mathrm{End}(V_i)$$
$$(B, I, J) \longmapsto \bigoplus_i \Big(\sum \varepsilon(h) B_h B_{\bar h} + I_i J_i \Big)$$

"Rough" definition of quiver variety

$$\mu^{-1}(0)/G_V \qquad G_V = \prod GL(V_i)$$

The quotient space $\mu^{-1}(0)/G_V$ does not have a structure of a variety (even a scheme).

Remark Ringel, Lusztig, ... did not consider quotient spaces

Solution 1 affine quotient

$$\mu^{-1}(0)/\!\!/G_V = \operatorname{Spec} \mathbb{C}[\mu^{-1}(0)]^{G_V} =: \mathfrak{M}_0(V,W)$$

$\underset{\text{as a set}}{=}$ isomorphism classes "semisimple framed representations"

Solution 2 GIT quotient

(B,I,J) : stable $\overset{\text{def.}}{\Longleftrightarrow}$ $\nexists\, 0 \neq S \subset V$ Q_0-graded subspace s.t. $J(S)=0$, $B(S) \subset S$

$$\{(B,I,J) \in \mu^{-1}(0) \mid \text{stable}\}/G_V =: \mathfrak{M}(V,W)$$

Remark There are more general definition : ζ-stable

$\zeta \in \operatorname{Hom}(\mathbb{Z}^{Q_0}, \mathbb{Q})$

<u>Example</u> A_1 $\mathfrak{g} = \mathfrak{sl}_2$

$$\begin{array}{c} V \\ {\scriptstyle J}\downarrow\uparrow{\scriptstyle I} \\ W \end{array} \qquad \mu(I, J) = IJ = 0$$

$$\mathfrak{M}_0(V,W) = \operatorname{Spec}(\mu^{-1}(0))^{G_V} \hookrightarrow \{X \in \operatorname{End}(W) \mid X^2 = 0\}$$

$$(I, J) \longmapsto X = JI \qquad \operatorname{rk} X \leq \dim V$$

stable $\Leftrightarrow$ J is injective $\quad \therefore \mathfrak{M}(V,W) \cong T^*Gr(\dim V, \dim W)$

(Grassmann manifold)

In particular $\mathfrak{M} = \phi$ if $\dim V > \dim W$

<u>Observation</u>

$$H_{\text{middle}}(\mathfrak{M}(V,W)) = \begin{cases} \mathbb{C} & 0 \leq \dim V \leq \dim W \\ 0 & \text{otherwise} \end{cases}$$

This is the same as weight spaces of

$L(\dim W)$: finite dimensional irreducible representation of $\mathfrak{sl}_2$ with h.w $= \dim W$

This is a special case of the main thm in the 2nd lecture.

Example 2

A_2

$\mathfrak{g} = \mathfrak{sl}_3$

$$\begin{tikzcd} V_1 \arrow[r, shift left] & V_2 \arrow[l, shift left, "B_h"] \\ \mathbb{C} \arrow[u, shift right, "I_1"'] \arrow[from=1-1, to=2-1, shift right, "J_1"'] & \end{tikzcd}$$

stability $\Rightarrow$ J_1 : injective

B_h : injective

$\therefore$ Three possibilities

$V_1 = V_2 = 0 \longrightarrow \mathfrak{M}(V, W) = pt$

$V_1 = \mathbb{C}$, $V_2 = 0 \longrightarrow$
$$\begin{tikzcd} \mathbb{C} & 0 \\ \mathbb{C} \arrow[u, shift right, dashed, "I_1 = 0"'] \arrow[from=1-1, to=2-1, shift right] & \end{tikzcd}$$
$\therefore \mathfrak{M}(V, W) = pt$

$V_1 = V_2 = \mathbb{C}$
$$\begin{tikzcd} \mathbb{C} \arrow[r, dashed, shift left, "= 0"] & \mathbb{C} \arrow[l, shift left] \\ \mathbb{C} \arrow[u, shift right, dashed, "= 0"'] \arrow[from=1-1, to=2-1, shift right] & \end{tikzcd}$$
$\mathfrak{M}(V, W) = pt$

$$H_{\text{middle}}(\mathfrak{M}(V, W)) = \begin{cases} \mathbb{C} & \text{three cases} \\ 0 & \text{otherwise} \end{cases}$$

This is the same as weight spaces of $\mathbb{C}^3$ as a representation of $\mathfrak{sl}_3$

irreducible, highest weight $= (1, 0)$