

Introduction to quiver varieties

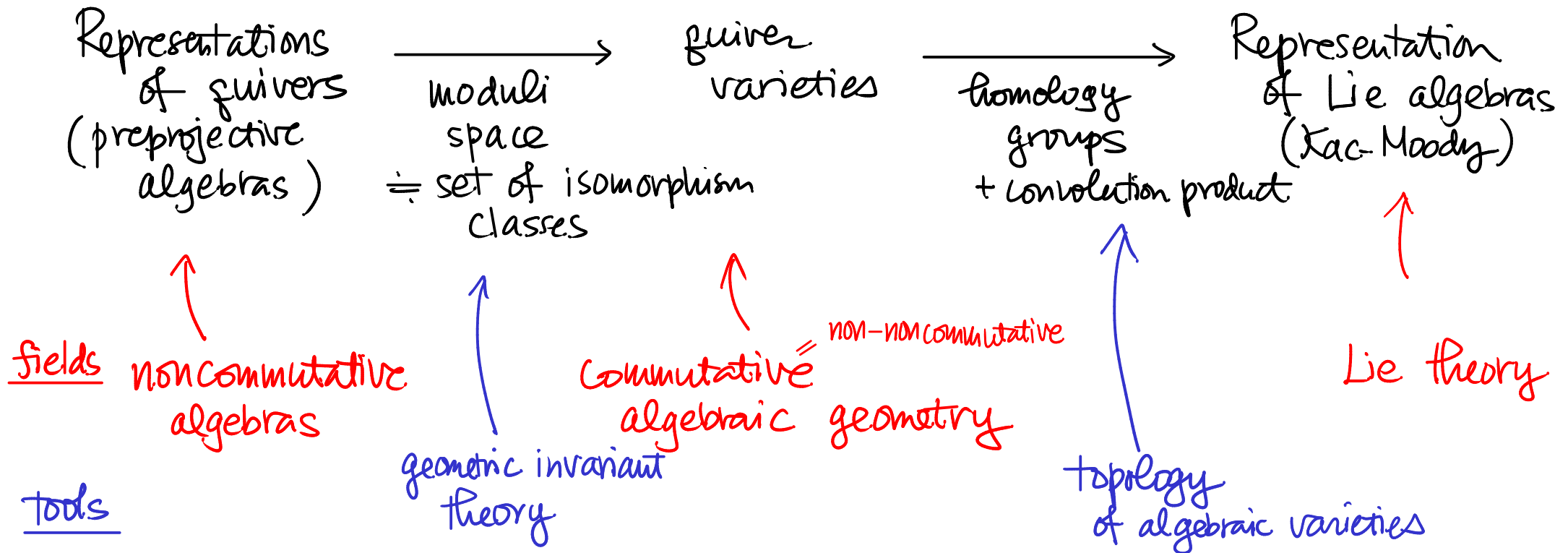
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<http://www.kurims.kyoto-u.ac.jp/~nakajima/Yotei.html>
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Connecting representation theories of quivers and Lie algebras



There are several other links.

- Ringel-Hall algebra \rightarrow realization of $U_q(\mathfrak{n}^-)$
 - Lusztig's canonical base
 - constructible functions on $\Lambda(V)$ (Lusztig)
- quantum enveloping algebra } motivating the above link

more recent development (motivated by earlier works of Lascoux-Ledez-Tubon Anki)

o quiver Hecke algebra (Khovanov-Lauda, Rouquier)

quiver \rightarrow a family of algebras $A_V \rightarrow$ categorification of representation
mod A_V of Lie algebras
 $+ (A_V, A_{V'})$ -bimodule

similar
construction

$D^b \text{Coh}(\text{quiver varieties}) \rightarrow$ larger categorification?

Notations

quiver $Q = (Q_0, Q_1)$

$$o(h) \xrightarrow{h} i(h)$$

$\overline{Q}_1 =$ opposite arrows

$$o(\overline{h}) \xleftarrow{\overline{h}} i(\overline{h})$$

$\begin{matrix} \text{"}i(\overline{h})\text{"} & & \text{"}o(\overline{h})\text{"} \end{matrix}$

extend --- to $Q_1 \cup \overline{Q}_1$ by $\overline{\overline{h}} = h$

space of representations

$$V = \bigoplus_{i \in Q_0} V_i \quad : \quad Q_0\text{-graded vector space } / \mathbb{C}$$

$$N(V) := \bigoplus_{h \in Q_1} \text{Hom}(V_{o(h)}, V_{i(h)}) \quad \hookrightarrow G_V = \prod \text{GL}(V_i)$$

$N(V)/G_V \xleftrightarrow{\text{bijective}}$ isomorphism classes of representations of Q with $\vec{\dim} = (\dim V_i)_{i \in Q_0}$

cotangent space

$$M(V) = N(V) \oplus N(V)^* = \bigoplus_{h \in Q_1 \cup \overline{Q}_1} \text{Hom}(V_{o(h)}, V_{i(h)})$$

moment map

$$\mu : M(V) \longrightarrow \bigoplus_i \text{End } V_i \quad ; \quad (B_h)_{h \in Q_1 \cup \overline{Q}_1} \longmapsto \left(\sum_{\substack{i(h)=i \\ \pm 1}} \varepsilon(h) B_h B_{\overline{h}} \right)_i$$

(preprojective alg.)

Remark

This has an origin in symplectic geometry

Lusztig's lagrangian $\Lambda(\mathcal{T}) = \{ (B_a) \mid \mu=0, \text{ nilpotent} \}$

Fact (Lusztig, Kashiwara-Saito) If ~~\mathcal{O}~~ , $\Lambda(\mathcal{T}) \subset M(\mathcal{T})$ is lagrangian, and $\# \text{Irr} \Lambda(\mathcal{T}) = \dim \mathcal{U}(\mathcal{T})_{\omega_{\mathcal{T}}} = -\dim \mathcal{T}$

framed representation of double quiver

$\mathcal{V}, \mathcal{W} : \mathbb{Q}_0$ -graded vector space / \mathbb{C}

$$M(\mathcal{V}, \mathcal{W}) = \bigoplus_{a \in \mathbb{Q}_1 \cup \overline{\mathbb{Q}}_1} \text{Hom}(\mathcal{V}_{0(a)}, \mathcal{V}_i(a_i)) \oplus \bigoplus_{i \in \mathbb{Q}_0} \text{Hom}(\mathcal{W}_i, \mathcal{V}_i) \oplus \text{Hom}(\mathcal{V}_i, \mathcal{W}_i)$$

\uparrow \uparrow \uparrow
 Denote components by B_a I_i J_i

← new part

Moment map

$$\mu : M(\mathcal{V}, \mathcal{W}) \rightarrow \bigoplus \text{End}(\mathcal{V}_i)$$

$$(B, I, J) \mapsto \bigoplus_i \left(\sum \varepsilon(a) B_a B_{\bar{a}} + I_i J_i \right)$$

"Rough" definition of quiver variety

$$\underline{\mu^{-1}(0)} / G_V$$

$$G_V = \prod \text{GL}(\mathcal{V}_i)$$

The quotient space $\underline{\mu^{-1}(0)}/G_V$ does not have
 a structure of a variety (even a scheme).

Remark Ringel, Lusztig, ... did **not** consider **quotient** spaces

Solution 1 affine quotient

$$\underline{\mu^{-1}(0)}/G_V = \text{Spec } \mathbb{C}[\underline{\mu^{-1}(0)}]^{G_V} =: \mathcal{M}_0(V, W)$$

as a set = isomorphism classes "semisimple framed representations"

Solution 2 GIT quotient

$$(B, I, J) : \text{stable} \stackrel{\text{det.}}{\iff} \neq_0 S \subset V \quad \begin{array}{l} \mathbb{Q}_0\text{-graded subspace} \\ \text{st.} \quad J(S) = 0, B(S) \subset S \end{array}$$

$$\{(B, I, J) \in \underline{\mu^{-1}(0)} \mid \text{stable}\} / G_V =: \mathcal{M}(V, W)$$

Remark There are more general definition : ζ -stable
 $\text{Hom}^n(\mathbb{Z}^{\mathbb{Q}_0}, \mathbb{Q})$

Example A_1 \mathfrak{sl}_2 $\begin{matrix} \mathbb{V} \\ \mathbb{J} \downarrow \uparrow \mathbb{I} \\ \mathbb{W} \end{matrix}$ $\mu(\mathbb{I}, \mathbb{J}) = \mathbb{I}\mathbb{J} = 0$

$$M_0(V, W) = \text{Spec}(\mu^{-1}(0))^{Gr} \hookrightarrow \{X \in \text{End}(W) \mid X^2 = 0\}$$

$$(\mathbb{I}, \mathbb{J}) \longmapsto X = \mathbb{J}\mathbb{I} \quad \text{rk } X \leq \dim V$$

stable $\Leftrightarrow \mathbb{J}$ is injective $\therefore M(V, W) \cong T^*Gr(\dim V, \dim W)$
 (Grassmann manifold)
 In particular $M = \emptyset$ if $\dim V > \dim W$

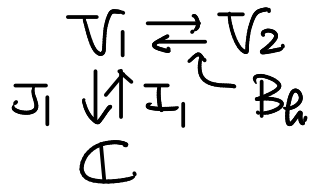
Observation $H_{\text{middle}}(M(V, W)) = \begin{cases} \mathbb{C} & 0 \leq \dim V \leq \dim W \\ 0 & \text{otherwise} \end{cases}$

This is the same as weight spaces of

$L(\dim W)$: finite dimensional irreducible representation of \mathfrak{sl}_2 with h.w = $\dim W$

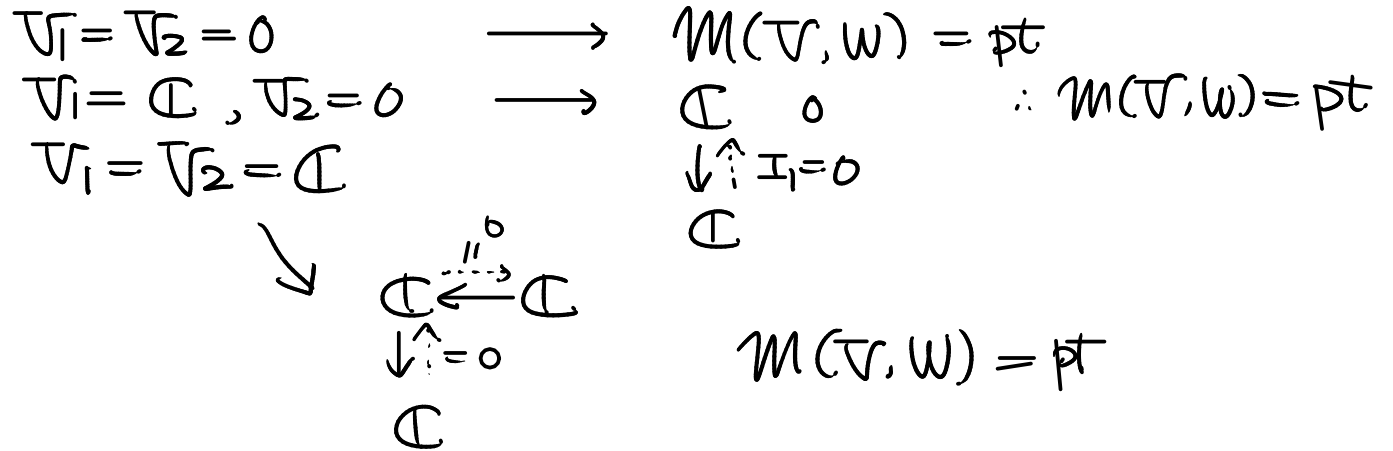
This is a special case of the main theorem in the 2nd lecture.

Example 2
 A_2
 $\mathfrak{g} = \mathfrak{sl}_3$



stability \Rightarrow J_1 : injective
 B_α : injective

\therefore Three possibilities



$$H_{\text{middle}}(M(\mathcal{V}, W)) = \begin{cases} \mathbb{C} & \text{three cases} \\ 0 & \text{otherwise} \end{cases}$$

This is the same as weight spaces of \mathbb{C}^3 as a representation of \mathfrak{sl}_3
irreducible, highest weight = (1, 0)